The Review of Contemporary Scientific and Academic Studies
An International Multidisciplinary Online Journal
www.thercsas.com

ISSN: 2583-1380 Vol. 3 | Issue No. 4 | April 2023 Impact Factor: 4.736 (SJIF)

Utilizing the Upadhyaya Transform to Solve the Linear Second Kind Volterra Integral Equation

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 Crossref/DOI: https://doi.org/10.55454/rcsas.3.04.2023.007

Abstract: *Volterra integral equations have a wide range of applications in mechanics, linear visco-elasticity, renewal theory, particle size statistics, damped vibration of a string, heat transfer problems, geometric probability, population dynamics, and epidemic studies. Many mathematicians and scientists are interested in finding either approximate or exact solutions to these equations. Upadhyaya Transform (UT) will be used in this paper to solve linear second type V.I.E. To accomplish this, the linear second type V.I.E. kernel has adopted a convolution type kernel. Some numerical examples are taken into account in order to outline the entire process of arriving at the solution. According to our findings, the Upadhyaya Transform (UT) is a powerful tool for finding solutions to the Linear Second kind V.I.E.*

Keywords: Faltung Theorem, Inverse Upadhyaya Transform, Upadhyaya Transform, Volterra Integral Equation

1. Introduction

The Volterra integral equation can be used to formulate a number of challenging problems in mathematics, chemistry, biology, astrophysics, and mechanics, including the radiative energy transfer problem, the oscillation problem for strings and membranes, and the quantum mechanical problem of momentum representation. Aggarwal *et al.* (2018) solved Volterra integral equations of the first and second kinds using various integral transforms (Kamal transform, Mahgoub transform, Mohand transform, Aboodh transform, and Elzaki transform). Upadhyaya (2019) introduces the Upadhyaya Integral Transform, a new generalization of the classical Laplace Transform. Mousa (2021) uses the Upadhyaya transform to solve the Volterra integral equation of the first kind. Kumar *et al.* (2022) created a new integral transform called the Rishi Transform and solved first kind Volterra integral equations. Patil *et al*. (2022) applied the Faltung theorem to the Kushare transform and its applications in first kind Faltung type Volterra integral equations. Patil *et al.* (2022) solve Volterra Integro-Differential equations of the first kind using the Kushare transform. Aggarwal *et al.* (2023) used the Rishi Transform to determine the analytical solution of a linear second kind V.I.E.

In this paper we demonstrate the applicability and significance of the Upadhyaya Transform (UT) in solving second type Linear Volterra integral equations.

2. Definition of Upadhyaya Integral Transform (Mousa, 2019)

Upadhyaya transformation of the function $f(t), t \geq 0$ is given by

$$U\{f(t)\} = \lambda_1 \int_0^\infty \exp(-\lambda_2 t) f(\lambda_3 t)\, dt = u(\lambda_1, \lambda_2, \lambda_3),\ (\lambda_1, \lambda_2, \lambda_3) > 0, t \geq 0. \quad (1)$$

The operator U is called the Upadhyaya Transform operator.

3. Inverse of Upadhyaya Transform (Mousa, 2019)

The inverse Upadhyaya transform of $u(\lambda_1, \lambda_2, \lambda_3)$, designated by $U^{-1}[u(\lambda_1, \lambda_2, \lambda_3)]$, another function $f(t)$ having the property $U\{f(t)\} = u(\lambda_1, \lambda_2, \lambda_3)$.

Tables (1-3) summarize some useful operational characteristics of the Upadhyaya Transform, as well as Upadhyaya Transforms and inverse Upadhyaya Transforms of some fundamental functions.

ISSN: 2583-1380 Vol. 3 | Issue No. 4 | April 2023 Impact Factor: 4.736 (SJIF)

Table 1: Some operational characteristics of Upadhyaya Transform (UT) (Mousa, 2019)

Name of Characteristics	Mathematical Form
Linearity	$U[b_1 f_1(t)+b_2 f_2(t);\lambda_1,\lambda_2,\lambda_3]=b_1 U[f_1(t);\lambda_1,\lambda_2,\lambda_3]+b_2 U[f_2(t);\lambda_1,\lambda_2,\lambda_3]$ $=b_1 u_1(\lambda_1,\lambda_2,\lambda_3)+b_2 u_2(\lambda_1,\lambda_2,\lambda_3)$, where b_1, b_2 are arbitrary constants.
Change of Scale	$U[f(t);\lambda_1,\lambda_2,\lambda_3]=u(\lambda_1,\lambda_2,\lambda_3)$ then $U[f(at);\lambda_1,\lambda_2,\lambda_3]$ $=U[f(t);\frac{\lambda_1}{a},\frac{\lambda_2}{a},\lambda_3]=u\left[\frac{\lambda_1}{a},\frac{\lambda_2}{a},\lambda_3\right]$
Translation	$U[f(t);\lambda_1,\lambda_2,\lambda_3]=u(\lambda_1,\lambda_2,\lambda_3)$ then $U[e^{at} f(t);\lambda_1,\lambda_2,\lambda_3]$ $=U[f(t);\lambda_1,\lambda_2-a\lambda_3,\lambda_3]=u[\lambda_1,\lambda_2-a\lambda_3,\lambda_3]$
Convolution	$U[f_1(t)*f_2(t);\lambda_1,\lambda_2,\lambda_3]=\frac{\lambda_3}{\lambda_1}u_1(\lambda_1,\lambda_2,\lambda_3)\cdot u_2(\lambda_1,\lambda_2,\lambda_3)$. Where $f_1(t)*f_2(t)$ is defined by $f_1(t)*f_2(t)=\int_0^t f_1(t-x)f_2(x)\,dx=\int_0^t f_1(x)f_2(t-x)\,dx$.

Table 2: Some fundamental function of Upadhyaya Transform (UT) (Mousa, 2019)

$f(t)$	$U[f(t);\lambda_1,\lambda_2,\lambda_3\}]=u(\lambda_1,\lambda_2,\lambda_3)$	$f(t)$	$U[f(t);\lambda_1,\lambda_2,\lambda_3]=u(\lambda_1,\lambda_2,\lambda_3)$
1	$\frac{\lambda_1}{\lambda_2}$	$\sin(at)$	$\frac{a\lambda_1\lambda_3}{\lambda_2^2+a^2\lambda_3^2}$
t	$\frac{\lambda_1\lambda_3}{\lambda_2^2}$	$\cos(at)$	$\frac{\lambda_1\lambda_2}{\lambda_2^2+a^2\lambda_3^2}$
t^m, $m\in N$	$\frac{m!\lambda_1\lambda_3^m}{\lambda_2^{m+1}}$	$\sinh(at)$	$\frac{a\lambda_1\lambda_3}{\lambda_2^2-a^2\lambda_3^2}$
$\exp(at)$	$\frac{\lambda_1}{\lambda_2-a\lambda_3}$	$\cosh(at)$	$\frac{\lambda_1\lambda_2}{\lambda_2^2-a^2\lambda_3^2}$

Table 3: Inverse Upadhyaya Transform of some fundamentals functions (Mousa, 2019)

$u(\lambda_1,\lambda_2,\lambda_3)$	$f(t)=U^{-1}[u(\lambda_1,\lambda_2,\lambda_3)]$	$u(\lambda_1,\lambda_2,\lambda_3)$	$f(t)=U^{-1}[u(\lambda_1,\lambda_2,\lambda_3)]$
$\frac{\lambda_1}{\lambda_2}$	1	$\frac{a\lambda_1\lambda_3}{\lambda_2^2+a^2\lambda_3^2}$	$\sin(at)$
$\frac{\lambda_1\lambda_3}{\lambda_2^2}$	t	$\frac{\lambda_1\lambda_2}{\lambda_2^2+a^2\lambda_3^2}$	$\cos(at)$
$\frac{m!\lambda_1\lambda_3^m}{\lambda_2^{m+1}}$	t^m, $m\in N$	$\frac{a\lambda_1\lambda_3}{\lambda_2^2-a^2\lambda_3^2}$	$\sinh(at)$
$\frac{\lambda_1}{\lambda_2-a\lambda_3}$	$\exp(at)$	$\frac{\lambda_1\lambda_2}{\lambda_2^2-a^2\lambda_3^2}$	$\cosh(at)$

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4. Solving linear second type V.I.E. via Upadhyaya Transform (UT)

Fundamental form of Linear Second type V.I.E. is provided by (Wazwaz, A. M. (2011)):

$$h(x) = f(x) + \lambda \int_0^x k(x, s) h(s)\, ds \tag{2}$$

where,
$$\begin{cases} h(x) = \text{unknown function}, \\ f(x) = \text{known function}, \\ k(x, s) = \text{kernal of integral equation}, \\ \lambda = \text{non}-\text{zero parameter}. \end{cases}$$

In this work, we presuppose that the kernel of equation (2) is a kernel of the Faltung type, i.e. $k(x, s) = k(x - s)$ .Consequently, equation (2) has the following form:

$$h(x) = f(x) + \lambda \int_0^x k(x - s) h(s)\, ds \tag{3}$$

Applying the UT to equation (3) and using the Faltung theorem, we have

$$U[h(x);\lambda_1,\lambda_2,\lambda_3] = U[f(x);\lambda_1,\lambda_2,\lambda_3] + \lambda U\left[\int_0^x k(x-s)h(s)\,ds;\lambda_1,\lambda_2,\lambda_3\right]$$

$$\Rightarrow U[h(x);\lambda_1,\lambda_2,\lambda_3] = U[f(x);\lambda_1,\lambda_2,\lambda_3] + \lambda U[k(x);\lambda_1,\lambda_2,\lambda_3]U[h(x);\lambda_1,\lambda_2,\lambda_3]$$

$$U[h(x);\lambda_1,\lambda_2,\lambda_3] = U[f(x);\lambda_1,\lambda_2,\lambda_3] + \lambda\left(\frac{\lambda_3}{\lambda_1}\right)U[k(x);\lambda_1,\lambda_2,\lambda_3]U[h(x);\lambda_1,\lambda_2,\lambda_3]$$

After simple computations:

$$U[h(x);\lambda_1,\lambda_2,\lambda_3] = \frac{U[f(x);\lambda_1,\lambda_2,\lambda_3]}{\left[1 - \lambda\left(\frac{\lambda_3}{\lambda_1}\right)U[k(x);\lambda_1,\lambda_2,\lambda_3]\right]} \tag{4}$$

Inverting the Upadhyaya Transform (UT), we get

$$h(x) = U^{-1}\left\{\frac{U[f(x);\lambda_1,\lambda_2,\lambda_3]}{\left[1 - \lambda\left(\frac{\lambda_3}{\lambda_1}\right)U[k(x);\lambda_1,\lambda_2,\lambda_3]\right]}\right\} \tag{5}$$

5. Numerical Applications

The entire procedure of determining the solution of linear second type V.I.E. with Faltung type kernel is explained in this section using four numerical examples.

(A) Consider the following second type of Linear V.I.E. with a Faltung-type kernel (Aggarwal *et al.* 2023):

$$h(x) + \int_0^x h(s)\, ds = \cos x + \sin x \tag{6}$$

Applying the UT to equation (6) and using the Faltung theorem, we have

$$U[h(x);\lambda_1,\lambda_2,\lambda_3] + U\left[\int_0^x h(s)\,ds;\lambda_1,\lambda_2,\lambda_3\right] = U[\cos x;\lambda_1,\lambda_2,\lambda_3] + U[\sin x;\lambda_1,\lambda_2,\lambda_3]$$

$$U[h(x);\lambda_1,\lambda_2,\lambda_3] + U[1;\lambda_1,\lambda_2,\lambda_3]U[h(x);\lambda_1,\lambda_2,\lambda_3] = U[\cos x;\lambda_1,\lambda_2,\lambda_3] + U[\sin x;\lambda_1,\lambda_2,\lambda_3]$$

$$U[h(x);\lambda_1,\lambda_2,\lambda_3] + \frac{\lambda_3}{\lambda_1}\times\left(\frac{\lambda_1}{\lambda_2}\right)U[h(x);\lambda_1,\lambda_2,\lambda_3] = \frac{\lambda_1\lambda_2}{(\lambda_2^2+\lambda_3^2)} + \frac{\lambda_1\lambda_3}{(\lambda_2^2+\lambda_3^2)} \tag{7}$$

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After simple computations:

$$U[h(x);\lambda_1,\lambda_2,\lambda_3]=\frac{\lambda_1\lambda_2^2}{(\lambda_2^2+\lambda_3^2)(\lambda_2+\lambda_3)}+\frac{\lambda_1\lambda_2\lambda_3}{(\lambda_2^2+\lambda_3^2)(\lambda_2+\lambda_3)}$$

$$\Rightarrow U[h(x);\lambda_1,\lambda_2,\lambda_3]=\frac{\lambda_1\lambda_2}{(\lambda_2^2+\lambda_3^2)} \quad (8)$$

We obtain the required solution of equation (6) by inverting equation (8).

$$h(x)=U^{-1}\left[\frac{\lambda_1\lambda_2}{(\lambda_2^2+\lambda_3^2)}\right]\Rightarrow h(x)=\cos x$$

(B) Consider the following second type of Linear V.I.E. with a Faltung-type kernel (Aggarwal *et al.* 2023):

$$h(x)+\int_0^x (x-s)\,h(s)\,ds=x \quad (9)$$

Applying the UT to equation (9) and using the Faltung theorem, we have

$$U[h(x);\lambda_1,\lambda_2,\lambda_3]+U\left[\int_0^x (x-s)h(s)\,ds;\lambda_1,\lambda_2,\lambda_3\right]=U[x;\lambda_1,\lambda_2,\lambda_3]$$

$$U[h(x);\lambda_1,\lambda_2,\lambda_3]+\frac{\lambda_3}{\lambda_1}U[x;\lambda_1,\lambda_2,\lambda_3]U[h(x);\lambda_1,\lambda_2,\lambda_3]=\frac{\lambda_1\lambda_3}{\lambda_2^2}$$

$$U[h(x);\lambda_1,\lambda_2,\lambda_3]+\frac{\lambda_3}{\lambda_1}\left[\frac{\lambda_1\lambda_3}{\lambda_2^2}\right]U[h(x);\lambda_1,\lambda_2,\lambda_3]=\frac{\lambda_1\lambda_3}{\lambda_2^2} \quad (10)$$

After simple computations:

$$U[h(x);\lambda_1,\lambda_2,\lambda_3]=\left[\frac{\lambda_1\lambda_3}{\lambda_2^2+\lambda_3^2}\right] \quad (11)$$

We obtain the required solution of equation (9) by inverting equation (11).

$$h(x)=U^{-1}\left[\frac{\lambda_1\lambda_3}{\lambda_2^2+\lambda_3^2}\right]=\sin x$$

(C) Consider the following second type of Linear V.I.E. with a Faltung-type kernel (Aggarwal *et al.* 2023):

$$h(x)+\int_0^x (x-s)\,h(s)\,ds=1-x \quad (12)$$

Applying the UT to equation (12) and using the Faltung theorem, we have

$$U[h(x);\lambda_1,\lambda_2,\lambda_3]+U\left[\int_0^x (x-s)h(s)\,ds;\lambda_1,\lambda_2,\lambda_3\right]=U[1;\lambda_1,\lambda_2,\lambda_3]-U[x;\lambda_1,\lambda_2,\lambda_3]$$

$$U[h(x);\lambda_1,\lambda_2,\lambda_3]+\frac{\lambda_3}{\lambda_1}U[x;\lambda_1,\lambda_2,\lambda_3]U[h(x);\lambda_1,\lambda_2,\lambda_3]=U[1;\lambda_1,\lambda_2,\lambda_3]-U[x;\lambda_1,\lambda_2,\lambda_3]$$

$$\Rightarrow U[h(x);\lambda_1,\lambda_2,\lambda_3]+\frac{\lambda_3}{\lambda_1}\left[\frac{\lambda_1\lambda_3}{\lambda_2^2}\right]U[h(x);\lambda_1,\lambda_2,\lambda_3]=\frac{\lambda_1}{\lambda_2}-\frac{\lambda_1\lambda_3}{\lambda_2^2} \quad (13)$$

After simple computations:

$$U[h(x);\lambda_1,\lambda_2,\lambda_3]=\left[\frac{\lambda_1\lambda_2}{\lambda_2^2+\lambda_3^2}\right]-\left[\frac{\lambda_1\lambda_3}{\lambda_2^2+\lambda_3^2}\right] \quad (14)$$

We obtain the required solution of equation (12) by inverting equation (14).

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$$h(x) = U^{-1}\left[\frac{\lambda_1\lambda_2}{\lambda_2^2+\lambda_3^2}\right] - U^{-1}\left[\frac{\lambda_1\lambda_3}{\lambda_2^2+\lambda_3^2}\right]$$

$$\Rightarrow h(x) = \cos x - \sin x$$

(D) Consider the following second type of Linear V.I.E. with a Faltung-type kernel (Aggarwal *et al.* 2023):

$$h(x) - \int_0^x h(s)\, ds = x \tag{15}$$

Applying the UT to equation (15) and using the Faltung theorem, we have

$$U[h(x);\lambda_1,\lambda_2,\lambda_3] - U\left[\int_0^x h(s)\,ds;\lambda_1,\lambda_2,\lambda_3\right] = U[x;\lambda_1,\lambda_2,\lambda_3]$$

$$U[h(x);\lambda_1,\lambda_2,\lambda_3] - \frac{\lambda_3}{\lambda_1}U[1;\lambda_1,\lambda_2,\lambda_3]U[h(x);\lambda_1,\lambda_2,\lambda_3] = U[x;\lambda_1,\lambda_2,\lambda_3]$$

$$U[h(x);\lambda_1,\lambda_2,\lambda_3] - \frac{\lambda_3}{\lambda_1}.\frac{\lambda_1}{\lambda_2}U[h(x);\lambda_1,\lambda_2,\lambda_3] = \frac{\lambda_1\lambda_3}{\lambda_2^2} \tag{16}$$

After simple computations:

$$U[h(x);\lambda_1,\lambda_2,\lambda_3] = \left[\frac{\lambda_1}{\lambda_2-\lambda_3}\right] - \left[\frac{\lambda_1}{\lambda_2}\right] \tag{17}$$

We obtain the required solution of equation (15) by inverting equation (17).

$$h(x) = U^{-1}\left[\frac{\lambda_1}{\lambda_2-\lambda_3}\right] - U^{-1}\left[\frac{\lambda_1}{\lambda_2}\right]$$

$$\Rightarrow h(x) = e^x - 1.$$

6. Conclusion

In this paper, we successfully used the Upadhyaya transform to obtain the solution of second type V.I.E. The outcomes demonstrate that Upadhyaya transform is a very effective integral transform for obtaining the exact solution of linear V.I.E of second type without needing to perform laborious and time-consuming computational work.

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