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Structured Linear Algebra

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Abstract: *Matrices with a special structure are omnipresent in computer algebra. These are matrices whose elements enjoy a certain repetition, or satisfy certain relationships. More formally, a structured n×n matrix is typically defined by elements O(n), instead of n^2 for a dense matrix, and can be multiplied by a vector in O˜(n) arithmetic operations, instead of operations O(n^2) for a dense matrix. This article presents a unified algorithm for handling dense structured matrices, such as Toeplitz, Hankel, Vandermonde and Sylvester matrices. Calculations with matrices of these classes are linked to calculations with polynomials, which allows the use of fast polynomial multiplication techniques to speed up their manipulation. For example, we can solve a linear system defined by an invertible structured matrix in O˜(n)operations.*

Keywords: Algebra, Algorithm, Computer Algebra, Matrices, , Polynomials, Structured Matrix

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1. Introduction

It is possible to manipulate the dense matrices of size$n \times n$to coefficients in a body $\mathbb{K}$in$\mathrm{MM(n)} = \mathrm{O}(n^{\theta})$ opérations in $\mathbb{K}$, where θis based on 2 and 3. Here, to handle, we intend to multiply, reverse, calculate the determinant, or solve a linear system. In some situations, the matrices that are handled in handles show a structure that these general algorithms do not know how to detect and exploit (Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest and Clifford Stein, 2009).

1.1. Definition

Amatrix $A \in M_n(\mathbb{K})$is called Toeplitz (or simply Toeplitz) if it is invariant along diagonals, that is to say if its elements$a_{i,j}$verification$a_{i,j} = a_{i+k,j+k}$for all$k$.

Such a matrix is completely defined by its first line and its first column.

1.2. Definition

A matrix $A \in M_n(\mathbb{K})$is called Hankel if sche is invariant along the ant-diagonal, that is to say if its elements$a_{i,j}$verified$a_{i,j} = a_{i-k,j+k}$for everything$k$.

Such a matrix iscompletrly denied by its first line and last column. (Bostan, Alin, Claude-Pierre Jeannerod et Éric Schost, 2008)

1.3. Example

The matrix of polynomial multiplication in fixe degree (in canonical bases) by a fixe polynoma is Toeplitz. There is also the specialized Toeplitz matrix, called tape. In fact, all Toeplitz matrix can be seen as a subdirator of a Toeplitz tape matrix, and this causes the next lemme (1.4 Lemme)**.**

1.4. Lemme

The product of a Toeplitz (or Hankel) matrix of$M_n(\mathbb{K})$by a vector$\mathbb{K}^n$can be done in$\mathrm{O(M(n))}$ operation.

Demonstration. For all $0 \leq i \leq n-1$, the element of the matrix-vector product.

$$\begin{pmatrix} a_{n-1} & \cdots & a_0 \\ \vdots & \ddots & \vdots \\ a_{2n-2} & \cdots & a_{n-1} \end{pmatrix} \times \begin{pmatrix} b_0 \\ \vdots \\ b_{n-1} \end{pmatrix} = \begin{pmatrix} c_0 \\ \vdots \\ c_{n-1} \end{pmatrix}$$

Is the coefficientX^{n-1+i} from the polynomium product

$$(a_0 + \cdots + a_{2n-2}X^{2n-2}) \times (b_0 + \cdots + b_{n-1}X^{n-1}).$$

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The evidence is quite similar for a Hankel matrix. Previous evidence shows that $2\mathrm{M}(\mathrm{n}) + \mathrm{O}(\mathrm{n})$ sufuses operations smell to multiply a matrix of Toeplitz or Hankel of size n by a vector. The terminal can be reduced to$\mathrm{M}(\mathrm{n}) + \mathrm{O}(\mathrm{n})$, observing that the extraction of the median part of the product is sufhi.

1.5. Example

A Sylvester matrix is the concatenation of two dish mattresses Toeplitz band.

1.6. Definition

A matrix $A = (a_{\mathrm{i,j}})_{\mathrm{i,j}=0}^{\mathrm{n}-1}$ de $M_n(\mathbb{K})$is called de Vandemonde if his elements are writing.

$a_{\mathrm{i,j}} = a_i^j$ to$a_0 + \cdots + a_{\mathrm{n}-1} \in \mathbb{K}$. (Pan, Victor Y. 2001)

1.7. Definition

A matrix $A = (a_{\mathrm{i,j}})_{\mathrm{i,j}=0}^{\mathrm{n}-1}$ de $M_n(\mathbb{K})$if called Cauchy if his elemnts are$a_{\mathrm{i,j}} = {}^{1}/_{(a_i - b_j)}$to $a_i,\ b_j \in \mathbb{K}$ avec $a_i \neq b_j$for alli, j.

There are a number of points in common between these

1.8. Examples:

The matrix is representative by elements$\mathrm{O}(\mathrm{n})$; the matrix-vector product may be faster than in the generic case, in $\mathrm{O}(\mathrm{M}(\mathrm{n})\mathrm{logn})$ operations instead of$\mathrm{O}(n^2)$; the by any matrix may be completed in almost the optimal complexity $\mathrm{O}(\mathrm{nM}(\mathrm{n})\mathrm{logn})$ in the size of the output for a polynomial multiplication based on FFT. Indeed, in each case, the matrix-vector$\mathrm{A}\mathcal{v}$ product admits an analytical interpretation:

— polynomial multiplication in the case Toeplitz, Hankel and Sylvester;

— Polynomial multipoint evaluation in the case of a Vandermondematrix;

— Multipoint valuation of rational fractions of the form $\sum_{j=0}^{n-1} c_j\,/(X - b_j)$in the case of a Cauchy matrix.

In view of these examples, we could therefore be tempted to define as structured a matrix such as its product by a vector can be done in$\tilde{O}(n)$operations. The question that is naturallybeing: is perhaps this definition of the structure of A to solve the system$\mathrm{Ax} = \mathrm{b}$? A first positive finding is that in each of the previous examples, the resolution also admits an analytical interpretation:

i. Recovering grocery of coefficients censive constants, if A of Toeplitz or Hankel;

ii. Polynomial interpolation, if Ais Vandermonde;

iii. Interpolationof rational fractions,if Ais Cauchy.

Indeed, if the following (a_n) of elements of $\mathbb{K}$verifying a recurrence (unknown) to coefficients constant frequently the shape $a_{n+d} = p_{d-1}a_{n+d-1} + \cdots + p_0 a_n,\ n \geq 0$,so find the coefficients $p_i$of this recurrence is to solve the Hankel system.

Another encouraging point is that for these three operations, almost alternatives algorithms that are almost optimal$\mathrm{O}(\mathrm{M}(\mathrm{n})\mathrm{logn})$. There are, however, some negative points: a first is that if Ait is an invertible matrix such as the linear application $\mathcal{v} \rightarrow A\mathcal{v}$calculate in L opérations, this does not imply the existence of an algorithm of complexity $\mathrm{O}(\mathrm{L})$ for the application $\mathcal{v} \rightarrow A^{-1}\mathcal{v}$. In other words, there is a lack of a reliability principle analogous to the transposition principle. For example, for hollow matrices, the best resolution algorithm, due to Wiedemann, is quadratic complexity in$n$. A second negative point is that the classes views above are not stable by reversal: the reverse of a dress of Toeplitz (Vandermonde) is not Toeplitz (Vandermonde). It is therfore necessary to add hypotheses in the definition of the correct concept of structured matrix. This definition exploits the generalization of the diagonal invariant character of a Toeplitz matrix, and comes from simple observation: if$A$ is to Toeplitz, then the matrix $\varphi(\mathrm{A}) = \mathrm{A} - (\mathrm{A\ décalée\ de\ 1\ vers\ le\ bas\ et\ de\ 1\ vers\ la\ droite})$ admits a nurture square sub-matrix of less than one; it is therefore ranked no more than one (we agreed to fill with zeros the first line and column of the shfted matrix). It is said that φ a moving operator and the movement of movement over to be more than.

It can be represented φ(A) in compact form, like a product$G.^t H$, withG et H rectangular size mattersn × 2. (Morf, M. 1980)

1.9. Example

If A is the matrix3 × 3

$$\begin{pmatrix} c & d & e \\ b & c & d \\ a & b & c \end{pmatrix}, avec\ d \neq 0, so\ \varphi(A)\ written$$

$$\varphi(A) = \begin{pmatrix} c & d & e \\ b & c & d \\ a & b & c \end{pmatrix} = \begin{pmatrix} c & d \\ b & 0 \\ a & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e/d \end{pmatrix}$$

1.10. Definition

The matrices (G, H)de $M_{n,2}$ (𝕂)telles que φ(A) = $G.^t$ Hare called motion generators for the Toeplitz A matrix.

These definitions extend to any matrix and lead to the concept of quasi-Toeplitz matrix.

1.11. Definition

More generally displacement operatorφ_+application

$A \to A - Z.A.^t Z$, where$A$is any matrix and Z the matrix define by

$$Z = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \vdots\cdots & 0 \\ \vdots & \ddots & \cdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}.$$

ZA is the matrix A shifted from 1 line down, et $A.^t Z$is the matrix$A$shifted from 1 column to the right, so well found the previous denial for dish of Toeplitz.The rankof emerging movement$A$as previously the whole α_+(A) = rang(φ + (A)).

It is also called a movement generator for the operatorφ_+ a couple(G, H) of matrices n × αverial equalityφ_+(A) = $G.^t$ H. Si α_+(A) ≪ n, It is said that A is quasi-Toeplitz>>. The whole α_+(A) is called length of the generator(G, H). (Gohberg, I. et V. Olshevsky, 1994).

N.B. The index+is introduced to do during a subsequent index used later.

Intuitively, the rank of movement measures how the matrice A is far from being Toeplitz or a concatenation of Toeplitz.

1.12. Example

The Toeplitz and Sylvester matrices are quasi-Toeplitz, of travel deck 2. Another extension makes it possible to define quasi-vandarmond and quasi-cauchy matrices, but for different displacement operators.

Rapid algorithms for structured mark structured matrices is the use of travel generators as a compact αstructure of sizeO(αn), therefore proportional to the size of the matrix, in the case wher$\alpha \leq n$.(Pan, Victor 1990).

1.13. Theorem (Cadre quasi-Toeplitz, quasi-Vandermonde, quasi-Cauchy).

Either$\alpha$the rankof adequate displacement, Aa matrix ofM_n(𝕂)given by movement generators size n × α, and either$b$a vector of$\mathbb{K}^n$. So, it is possible to :

a) Calculate the determinant of$A$;

b) Calculate the rank of A;

c) Calculate a solution of the system Ax = b, or prove that there is no in $\tilde{O}(\alpha^2 n)$ operations in𝕂. More exactly, this complexity is expressed in terms of the polynomial multiplication function, and is worth:

1. $O(\alpha^2 M(n) \log n)$in thecas quasi-Toeplitz ;
2. $O(\alpha^2 M(n) log^2 n)$ in the almost quasi-Vandermondeand quasi-Cauchy.

This theorem has a constitutional uniberenic algorithmic to solve in almost complexity problems with the polynomials and series. (Pan, Victor Y. Et Ailong Zheng 2000).

1.14. Corollary

We can calculate in$O(M(n) \log n)$ arithmetic transactions:

1. The extended PGCD and the resulting of two polymades of degree bounded by n;

2. A romantic of pady type $(n, n)$of a primed truncated series$2n$.

Demonstration.

The resulting of two polynomials$A, B \in \mathbb{K}[X]$is getting as determining the Sylvester matrix$\mathrm{Syl}(A, B)$, who has a rank of movement over the position 2. The degree ofpgcd $(A, B)$is obtained by a calculation of the rank of thematrix of Sylvester:

$$\deg(G) = \deg(A) + \deg(B) - \operatorname{rank}(\mathrm{Syl}(A, B)).$$

Once known the degree of the PGCD, relationship of Bézout$UA + VB = G$,with the contraints$\deg(U) < deg(B) - deg(G)\ et\ deg(V) < deg(A) - deg(G)$, translates into a linnaer system in thecoefficientde U, V et G, whose matrix is a concatenation of three Toeplitz matrix, so a quasi-Toeplitz matrixof movement of more movement. Similar considerations apply for the calculation of approximate of Padé. (Cardinal, Jean-Paul, 1999).

1.15. Corollary

Being $n$series $f_1, \ldots, f_n$ of $\mathbb{K}[[X]]$ known to precision$\sigma = \sum_i (di + 1) - 1$, it is possible to calculate an approximation ofPadé–Hermite $(p_1, \ldots, p_n)$ of type $(d_1, \ldots, d_n)$ in $O(n^2 M(\sigma) \log\sigma)$ operationsin$\mathbb{K}$.

Demonstration.

This is a linear problem in the coefficient of the approximation. The matrix is almost in even it, from marked mock terminated by$n$.(Kailath, T., S. Y. Kung et M. Morf, 1979).

2. Quasi-Toeplitz Case

In the following, we will focus only on the case of quasi-Toeplitz because it contains the main algorithmic ideas, and it covers many of the applications; the quasi-vandarmond and quasi-cauchy cases are reluctant and are more technical. We will show that the concept of matrix quasi-Toeplitz is a good concept of structure, as it meets the following properties:

(P1) The product of a quasi-Toeplitz matrix is found, if not, and the vector is essentially the ester; (P2) the sum and the product of two quasi-Toeplitz matrices remain almost quasi-Toeplitz;

(P3) l'inverse auss. (Bitmead, Robert R. Et Brian D. O. Anderson 1980).

However, these properties form the minimum required to design a stere-reversed algorithm in the compact representation by movement generator.

3. Matrix-Vectorproduct, Quasi-Toeplitz Case

To show the property (P1) above, the key point is the following.

3.1. Proposition

Formula ΣLU. The operator$\varphi_+ : A \longmapsto A - Z.A.^t Z$isinvertible. More exactly we have the following formula, called formula ΣLU :

$A - Z.A.^t Z = G.^t H$if and only if$A = \sum_{i=1}^{\alpha} L(x_i).U(y_i)$, where the$x_i$ (resp. y_i)are the generator columns$G$ (resp. H), and where, for a column vector $v = ^t [v_0 \ldots v_{n-1}]$, we are noted$L(v)$ the folwer triangular Toeplitz matrix

$$\begin{pmatrix} v_0 & 0 & \cdots & 0 \\ v_1 & v_0 & \cdots & 0 \\ \vdots & \ddots & \cdots & \vdots \\ v_{n-1} & \cdots & v_1 & v_0 \end{pmatrix}$$

and U(v) the triangular sudden Toeplitz matrix ${}^tL(v)$.

Demonstration.

By linearity, he sufed to treat the cosα = 1. ifC = L(a)U(b), then an immediate calculation shows that

$c_{i,j} = a_i b_j + a_{i-1} b_{j-1} + \cdots$,and so$c_{i,j}$ - $c_{i-1,j-1} = a_i b_j$ et $\varphi_+(C) = (a_i b_j)_{i,j=0}^{n-1} = a.^t b$.

The reverse involvement is due to the injectivity ofφ_+.

(Brent, Richard P., Fred G. Gustavson Et David Y. Y. Yun, 1980. Gustavson, Fred G. Et David Y.Y. Yun, 1979).

3.2. Definition

The numberα_+(A) is the smallest whole arethere is a decomposition of the form.

$$A = \sum_{i=1}^{\alpha} L_i U_i$$

for matrices $L_i$of Toeplitz lower, and$U_i$of Toeplitz higher. (Levinson, Norman, 1947)

3.3. Example

If A is of Toeplitz, so by notingA_{inf}its lower part andA_{ssup}its strictly superior part, it is that$A$ admitsthe decomposition$A = A_{inf} . I_n + I_n . A_{ssup}$, so$\alpha_+$(A) ≤ 2 (Trench, William F. ,1964).

3.4. Corollary

If A is given in compact representation by a pair of movement generator(G, H) of size$n \times \alpha$, then the matrix-vector productAv can be performedO(αM(n))arithmetic operations.

Demonstration.

The productAv written as the sum ofα terms, all of the form$L(x_i)(U(y_i)v)$L(xi)(U(yi)v). however, each of these an be calculatedO(M(n)) operations. (Golub, Gene H. Et Charles F. Van Loan, 2013)

3.5. Proposition (Matrix operations in compact representation).

Either (T, U) length displacacement generatorα for A , and (G, H) generator of lengthβfor B. So

1. ([T|G], [U|H]) are displacement generators forA + B lengthα + β;

2. ([T|W|a], [V|H| − b]) are displacement generators for AB, length α + β + 1, où $V :=^t B.U, W := A.^t Z.G$, and where the vector a (resp. b) is the last column ofZ. A (resp. de $Z.^t B$). (Gohberg, I. C. Et A. A. Semencul, 1972)

3.6. Corollary

In compact representation by displacement generators, of lengths at most$\alpha$for Aet B, we can calculate

1The sumA + B inO(αn)operationsin𝕂 ;

2. The productAB inMul(n, α): = O(α^2M(n)) operations in 𝕂.

Demonstration.

The only non-trivial point is the calculation of the matricesV and W and vectorsa and b. Everything is resting on the formulaΣLU. IfBwritten$\sum_{i=1}^{\beta} L(x_i)U(y_i)$, then his transposwritten${}^tB = \sum_{i=1}^{\beta} L(y_i)U(x_i)$, and the calculation of$V = {}^tB$.U gets back to2αβpolynomial multiplications, each that may be figured in O(M(n)). The same reasoning applies to the calculation of W. finally, the calculationof $a$is returned to multiply A by the column^t[0, … ,0,1] and similar purpose for b allows you to conclude. (Kaltofen, Erich, 1994)

3.7. Definition

The operator$\varphi_-$of moving a matrix$A$of$M_n(\mathbb{K})$is defiated by equality

$\varphi_-(A) = A - {}^tZ.A.Z$ = A−(Ashifted from 1 up and left). The rankof travel$\alpha_-$is defined nom by the formula$\varphi_-(A) = \text{rang}(\varphi_-(A))$. We can show (in the same way as forφ_+) that is equivalent to the equity ofφ_-, and that$\varphi_+ and \varphi_-$are linked.(Kaltofen, Erich, 1995)

Conclusion

The first quadratic complexity algorithm for the resolution of a liner system whose matrix is of Toeplitz, symmetrical and positive deficiency. Subsequently, it has been shown that the opposite of such a matrix can be calculated in the same complexity.

The Trench balic algorithm is based on a formulaΣLU two terms of the reverse of matrix, the summaries being built from the first line and the first column of A^{-1}. Brent, Gustavsonand Yunshowed how to reduce the calculation of the first line and column of A^{-1}at a copound of approximation of padé. They also highlighted the deep links between resolution of Toeplitz systems, the extended algorithm and the approximation of padé, and proposed the first algorithm for the resolution of a system definition by wrestling inverse Toeplitz matrix sizen in O(M(n)logn) arithmeticsoperations.

At the same time, the concept of travel mark was introduced by Kailath, Kung and Morf, to be able to resolve a reverse quasi-Toeplitz system of travelαin cost O(αn^2).Quick versions of complexityO(α^d M(n)logn) were obtained independently by Bitmead and Anderson (withd $= 4$) and Morf (with d =2.Under assumptions of high regularity on entry. These assumptions were then lifted by Kaltofen. Quasi-Cauchy case extensions are due to cardinal. The Quasi-Vandermondecase is treated in Pan, and Gohberg and Olshevsky items. The best asymptotic complexity visualizes simultaneously on the size of the matrix and its travel mark is due to Bostan, Jeannerod and Schost, who found how to introduce rapid product of dense matrices in the algorithmic of structured matrices, and reduced the cost to$\tilde{O}(\alpha^{\theta-1}n)$for the resolution of the quasi-toeplitz, quasi-vandermond and quasi-cauchy systems. A good reference on the structured matrics is the Pan book.

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